Effect of Dissipation on Density Profile of One Dimensional Gas

Douglas A. Kurtze[1][*] and Daniel C. Hong[2][†]

[1] *Department of Physics, University of Arizona, Tucson, Arizona 85721*

and Department of Physics, North Dakota State University, Fargo, North Dakota 58105

[2] *Department of Physics, Lewis Laboratory, Lehigh University, Bethlehem, Pennsylvania 18015*

(February 1, 2008)

Abstract

We study the effect of dissipation on the density profile of a one-dimensonal gas that is subject to gravity. The gas is in thermal equilibrium at temperature T with a heat reservoir at the bottom wall. Perturbative analysis of the Boltzmann equation reveals that the correction due to dissipation resulting from inelastic collisions is positive for $0 \leq z < z_c$ and negative for $z > z_c$ with z the vertical coordinate. The numerically determined value for z_c is $mgz_c/k_BT \approx 1.1613$, where g is the gravitational constant and m is the particle mass.

I. INTRODUCTION

Granular materials are basically a collection of meso- to macroscopic particles that interact with each other via short range repulsive potentials [1]. For this reason, they may be considered as a very dense molecular gas, but with several key differences: first, since the particles are macroscopic, gravity plays an important role and cannot be dismissed, as it

---

[*]E-mail address: kurtze@golem.phys.ndsu.NoDak.edu

[†]E-mail address: dh09@lehigh.edu

can for a molecular gas. In fact, most of the unique features of granular materials disappear in the absence of gravity. Second, collisions among the particles are inelastic, so dissipation must be included in studies of the response of granular materials to external stimuli. It has recently been observed that dissipation appears to be responsible for the existence of a uniform cooling state and the associated clustering instability [2-7], and modifies the velocity distribution function from the usual Maxwellian to a power law or exponential function [8,9]. The question of whether the appearance of the clustering instability is a generic phenomenon or an artifact of the hard sphere potential remains open; nonetheless we note that the role of dissipation in nonequilibrium dynamics has always been subtle, complex, and mathematically challenging.

In this paper, we examine the effect of dissipation on the density profile of a one-dimensional granular gas within the framework of the Boltzmann equation. In so doing, we implicitly treat the grains as point particles, and thus ignore one of the crucial aspects of granular materials, namely the excluded volume effect [10,11]. Neverthless, the outcome of this investigation is interesting and requires further study because it makes a nontrivial prediction: the dissipation leads to an increase in density near the source and a decrease away from the source, with the dividing line appearing at a dimensionless height gz/T of approximately 1.1613. This may have some relevance to the recent observation of a clustering instability near the source in the presence of gravity [7]. However, it remains to be seen whether our result persists in higher dimensions, or survives the inclusion of the excluded volume effect.

II. PERTURBATIVE ANALYSIS OF THE ONE-DIMENSIONAL BOLTZMANN EQUATION

Consider a one-dimensional gas of N identical classical point particles moving along the z axis above a wall at $z = 0$, and acted upon by a constant gravitational field g in the negative z direction. The particles collide with one another inelastically, with a coefficient

of restitution $r < 1$. Let $N f(u, z)\, du\, dz$ be the average number of particles found between heights z and $z+dz$ with velocity between u and $u+du$, in the steady state. This distribution function is given by the one-dimensional Boltzmann equation with dissipation [2],

$$\left(u\frac{\partial}{\partial z} - g\frac{\partial}{\partial u}\right) f(u, z) = C(u, z; q) - A(u, z). \tag{1}$$

The collision terms on the right hand side are the "creation" term,

$$C(u, z; q) = \int\int |u' - u''|\, f(u', z)\, f(u'', z)\, \delta[u - qu' - (1 - q)u'']\, du'\, du'', \tag{2}$$

which gives the rate at which particles collide at z and come out of the collision with velocity u, and the "annihilation" term,

$$A(u, z) = f(u, z) \int |u' - u|\, f(u', z)\, du' = C(u, z; 0), \tag{3}$$

which is the rate at which particles with velocity u collide at z and emerge with some other velocity. The parameter q is related to the coefficient of restitution by $r = 1 - 2q$.

We assume that the bottom wall at $z = 0$ maintains the particles there in some given velocity distribution $f_b(u)$. This roughly means that the particle distribution f obeys the boundary condition $f(u, z = 0) = f_b(u)$ at the bottom wall, but we must be a bit more careful than this. Since the particles are not allowed into the region of negative z, the bottom wall can only propel particles upward. Particles at $z = 0+$ with positive velocities u must have been launched by the bottom wall, but those with negative velocities must have either undergone collisions or at least been in flight for a finite time since they were last in contact with the bottom wall. Therefore the velocity distribution of these latter particles is not $f_b(u)$, but instead is something which was established by the dynamics described by the Boltzmann equation itself. Only the former particles, then, those with positive velocities, must be in the velocity distribution imposed by the bottom wall. Thus the correct boundary condition is

$$f(u, 0) = f_b(u) \qquad \text{for} \qquad u \geq 0. \tag{4}$$

The other boundary conditions are the obvious ones, namely that f should vanish for $z \to \infty$ and for $|u| \to \infty$.

We now define a new parameter

$$\epsilon \equiv \frac{q}{1-q} = \frac{1-r}{1+r},$$ (5)

which is small when the collisions are only slightly dissipative, i.e., when the coefficient of restitution is only slightly less than unity. Writing the creation term (2) in terms of ϵ and carrying out the integral over u'' gives

$$C(u, z; q) = (1 + \epsilon)^2 \int |u' - u| \, f(u', z) \, f(u + \epsilon[u - u'], z) \, du'.$$ (6)

We now expand this in powers of ϵ and substitute it into the Boltzmann equation (1) to get

$$\left(u \frac{\partial}{\partial z} - g \frac{\partial}{\partial u} \right) f(u, z) = \epsilon \int |u' - u| \, f(u', z) \left[2 f(u, z) + (u - u') \frac{\partial f}{\partial u} \right] du' +$$

$$\epsilon^2 \int |u' - u| \, f(u', z) \left[f(u, z) + 2(u - u') \frac{\partial f}{\partial u} + \frac{1}{2}(u - u')^2 \frac{\partial^2 f}{\partial u^2} \right] du' + \cdots. \tag{7}$$

This is the starting point of our calculation.

Let us write the solution of (7) as a power series,

$$f(u, z) = f_0(u, z) + \epsilon f_1(u, z) + \epsilon^2 f_2(u, z) + \cdots$$ (8)

At zeroth order, f_0 must satisfy

$$u \frac{\partial f_0}{\partial z} - g \frac{\partial f_0}{\partial u} = 0.$$ (9)

This is satisfied identically by any function of the energy variable

$$E(u, z) = \frac{1}{2} u^2 + gz.$$ (10)

Thus to zeroth order we may satisfy the Boltmann equation and the boundary condition (5) by writing

$$f_0(u, z) = f_b(\sqrt{2E(u, z)}).$$ (11)

This is the complete solution to the problem in the case of elastic collisions. Note that there would have been *no* solution to this problem had we attempted to impose the boundary condition (5) for *all* velocities u, with a function f_b which is not even.

4

We will most often be interested in an assemblage which is in thermal equilibrium at $z = 0$. In this case we have

$$f_0(u, z) = \mathcal{N} \exp(-E/T) = \mathcal{N} \exp(-(u^2 + 2gz)/2T), \tag{12}$$

where the temperature is mT/k_B, where m is the particle mass and k_B is Boltmann's constant, and the normalization constant $\mathcal{N}$ is

$$\mathcal{N} = g/\sqrt{2\pi T^3}. \tag{13}$$

The zeroth order density profile is obtained by integrating over velocity,

$$\rho_0(z) = N \int_\infty^\infty f_0(u, z) du. \tag{14}$$

In the thermal equilibrium case (12), this becomes

$$\rho_0(z) = (Ng/T) \exp(-gz/T). \tag{15}$$

We now calculate the first-order correction f_1 to the distribution function. From (7), we see that this must satisfy

$$\left(u\frac{\partial}{\partial z} - g\frac{\partial}{\partial u} \right) f_1(u, z) = \int |u' - u| f_0(u', z) \left[2f_0(u, z) + (u - u')\frac{\partial f_0}{\partial u} \right] du'. \tag{16}$$

The right side of this equation is the u-derivative of $f_0(u, z) \int (u - u')|u - u'| f_0(u', z) \, du'$, but we have not found this intriguing fact to be very useful for our purposes. Note, however, that $f_0(u)$ is an even function of u, and therefore the right side of (16) is also even in u. Since the operator on the left side is odd in u, this implies that solving (16) will lead to a function which is odd in u. Since we find the density profile by integrating over all u, this function will not change the density from the zeroth order result (14). However, the operator in (16) also annihilates all functions of $E(u, z)$, so f_1 can also contain any function of E. Such a function is even in u, and so does contribute to the density profile.

For the thermal equilibrium case, we can calculate f_1 explicitly. First note that the z dependence of the right side of (16) consists entirely of a factor $\exp(-2gz/T)$ coming from the two f_0 factors in the integrand. To take advantage of this, we write f_1 in the form

$$f_1(u, z) = h_1(E(u, z)) - g^{-1}f_0^2(E(u, z))F_1(u). \tag{17}$$

Note that F_1 is independent of z. Substituting this expression, and the explicit form (12) of f_0, into (16) then gives

$$\frac{dF_1}{du} = \int_{-\infty}^{\infty} |u' - u| \left[2 + \frac{u(u' - u)}{T}\right] \exp\left(-\frac{u'^2 - u^2}{2T}\right) du'. \tag{18}$$

Changing integration variables from u' to $v = u' - u$ and doing some algebra, we find that this can be written as

$$\frac{dF_1}{du} = 2 \int_0^{\infty} v \left(4 - \frac{v^2}{T}\right) \exp\left(-\frac{v^2}{2T}\right) \cosh\left(\frac{uv}{T}\right) dv, \tag{19}$$

so we finally have

$$F_1(u) = 2T \int_0^{\infty} \left(4 - \frac{v^2}{T}\right) \exp\left(-\frac{v^2}{2T}\right) \sinh\left(\frac{uv}{T}\right) dv. \tag{20}$$

This can also be written out explicitly in terms of error functions.

We must now determine the function $h_1(E)$, which forms the even part of f_1. This is slightly more subtle than it first appears. The first thing we should do is apply the boundary condition (4) at $z = 0$, thus maintaining the thermal equilibrium distribution of particles coming upward from the bottom wall. Thus we would choose $h_1(E)$ to be $g^{-1} f_0^2(E) F_1(\sqrt{2E})$. However, as we will see below, this has a problem: the resulting density profile has fewer than N particles. In order to remedy this situation, we must change the normalization of f_0, or equivalently include an extra term proportional to $f_0(E)$ in $h_1(E)$, with a coefficient chosen to make the total number of particles in the system again equal to N. This amounts to replacing the boundary condition (4) with a slightly more general condition of the form

$$f(u, 0) \propto f_b(u) \qquad \text{for} \qquad u \geq 0, \tag{21}$$

with the proportionality constant chosen so that f is properly normalized. Thus we maintain the correct number of particles, and the *functional form* of the distribution of particles launched from the bottom wall. This relaxation of condition (4) is physically reasonable, because the normalization of the distribution f_b is not directly observable.

We now have an expression for the distribution function f correct to first order in the dissipation parameter ϵ,

$$f(u, z) = (1 + \epsilon \mathcal{N}_1) f_0(E) + \epsilon g^{-1} f_0^2(E) [F_1(\sqrt{2E}) - F_1(u)] + O(\epsilon^2), \tag{22}$$

6

where E is given by (10), f_0 by (12) and (13), F_1 by (20), and the constant $\mathcal{N}_1$ is yet to be determined. The density profile is obtained by integrating over u; since $F_1(u)$ is an odd function it drops out, leaving us with

$$N^{-1}\rho(z) = (1 + \epsilon\mathcal{N}_1)(g/T)\exp(-gz/T) + \epsilon g^{-1}\int f_0^2(E)F_1(\sqrt{2E})\,du + O(\epsilon^2). \qquad (23)$$

Integrating this equation over all z gives unity on the left side, so we see that $\mathcal{N}_1$ must be given by

$$\mathcal{N}_1 = -g^{-1}\int_0^\infty dz \int_{-\infty}^\infty du\, f_0^2(E)F_1(\sqrt{2E}). \qquad (24)$$

Remarkably, the integrals on the right can be evaluated analytically, giving the result

$$\mathcal{N}_1 = 2/\pi. \qquad (25)$$

Using this result in (23), we obtain the central result of this paper: an expression for the first-order correction to the density profile due to dissipation. After rescaling the integration variables u and v by factors of $T^{1/2}$, we get

$$N^{-1}\rho_1(z) = \frac{2}{\pi}\frac{g}{T}\left\{\exp(-gz/T)\right.$$

$$+ \exp(-2gz/T)\int_0^\infty du \int_0^\infty dv(4 - v^2)\exp(-u^2)\exp(-v^2/2)\sinh(v\sqrt{u^2 + 2(gz/T)})\Big\}. \qquad (26)$$

We have evaluated (26) numerically as a function of gz/T after first doing the integral over v analytically (in terms of error functions). The result is shown in Figure 1. The most important feature of $\rho_1(z)$ is the fact that it is *positive* near the source and *negative* away from the source, with a dividing line at $z_c = 1.1613T/g$. This is somewhat larger than T/g, which is the average z coordinate of the particles when their collisions are elastic. Thus below z_c the dissipation enhances the density profile, while above z_c it suppresses it. We might expect this behavior, since the dissipation in the collisions removes mechanical energy from the system. The effect of the thermal reservoir at the bottom wall is independent of the characteristics of the collisions, so the particles leaving the bottom wall are in the same distribution whether the collisions are elastic or not. However, when their collisions are inelastic, every collision reduces the total mechanical energy of the assemblage, thus

reducing the average height which the particles reach. This may have some relevance to the recent experimental observation of the clustering instability by Kudrolli et al. [7], who noticed the migration of the clustering instability along the direction of gravity and toward the driving source.

We note also that the perturbative correction to the density profile reaches a minimum of -0.206 at $gz/T \approx 2.22$, and it also has a small maximum of 1.0261 at $gz/T \approx 0.037$. We have not yet identified the physical mechanism responsible for the presence of this maximum.

It remains to be seen whether the enhancement of the density profile near the source in the presence of dissipation will persist in higher dimensions, or if it will survive the inclusion of excluded volume effects in the model. Future studies that elucidate these questions may provide interesting new insights into the role of dissipation in granular dynamics.

References

[1] H.M. Jaeger, S. R. Nagel, and R.P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).

[2] S. McNamara and W. R. Young, Phys. Fluids A **4**, 496 (1992); **5**, 34 (1993); S. McNamara, *ibid.* **5**, 3056 (1993).

[3] M. A. Hopkins and M. Y. Louge, Phys. Fluids A **3**, 4 (1990); I. Goldirsch and G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993).

[4] B. Bernu and R. Mazighi, J. Phys. A: Math. Gen. **23**, 5745 (1990).

[5] Y. Du, H. Li, and L.P. Kadanoff, Phys. Rev. Lett. **74**, 1268 (1995).

[6] E. E. Esipov and T. Poschel, J. Stat. Phys. **86**, 1385 (1997).

[7] A. Kudrolli, M. Wolpert, and J. P. Gollub, Phys. Rev. Lett. **78**, 1383 (1997).

[8] J. Brey, F. Moreno, and J.W. Duffty, Phys. Rev. E. **54**, 445 (1996).

[9] E. L. Grossman, T. Zhou, and E. Ben-Naim, preprint cond-matt/9607165

[10] H. Hayakawa and D. C. Hong, Phys. Rev. Lett. **78**, 2764 (1997).

[11] E. Clement and J. Rachjenbach, Europhys. Lett. **16**, 133 (1991).

Figure Captions

Fig. 1. The first-order correction $\rho_1(\tilde{z})$ to the density profile, given by (26), plotted as a function of the dimensionless height $\tilde{z} = gz/T$. The value at $\tilde{z} = 0$ is 1. The graph then rises to a small maximum at $\tilde{z} = 0.0372$ with a height of 1.0261, crosses zero at $\tilde{z} = 1.1613$, and reaches a minimum at $\tilde{z} = 2.2217$ with a value of -0.2057. For large $\tilde{z}$ it approaches the axis as $-2\tilde{z}\exp(-\tilde{z})$.